SEMI-INVARIANT $\xi^{\perp}$-SUBMANIFOLDS OF GENERALIZED QUASI-SASAKIAN MANIFOLDS

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Dedicated to the memory of Prof. Stere Ianuş (1939 − 2010)

Abstract. A structure on an almost contact metric manifold is defined as a generalization of well-known cases: Sasakian, quasi-Sasakian, Kenmotsu and cosymplectic. Then we consider a semi-invariant $\xi^{\perp}$-submanifold of a manifold endowed with such a structure and two topics are studied: the integrability of distributions defined by this submanifold and characterizations for the totally umbilical case. In particular we recover results of Kenmotsu [8], Eum [6] and Papaghiuc [12].

1. Preliminaries and basic formulae

An interesting topic in the differential geometry is the theory of submanifolds in spaces endowed with additional structures. In 1978, A. Bejancu (in [2]) studied CR-submanifolds in Kähler manifolds. Starting from it, several papers have been appeared in this field. Let us mention only few of them: a series of papers of B.Y. Chen (e.g. [5]), of A. Bejancu and N. Papaghiuc (e.g. [3] in which the authors studied semi-invariant submanifolds in Sasakian manifolds). See also [10]. The study was extended also to other ambient spaces, for example A. Bejancu in [4] also studied QR-submanifolds in quaternionic manifolds and M. Barros in [1] investigated CR-submanifolds in quaternionic manifolds. Several important results above CR-submanifolds are being brought together in [4], [5], [9], [10], [11] and the corresponding references. The purpose of the present paper is to investigate the semi-invariant $\xi^{\perp}$-submanifolds in a generalized Quasi-Sasakian manifold.

Let $\widetilde{M}$ be a real $(2n+1)$-dimensional smooth manifold endowed with an almost contact metric structure $(\phi, \xi, \eta, \tilde{g})$:

$$\begin{cases} \phi^2 = -I + \eta \otimes \xi, \ \eta(\xi) = 1, \ \eta \circ \phi = 0, \ \phi\xi = 0 \\ \eta(X) = \tilde{g}(X, \xi), \tilde{g}(\phi X, Y) + \tilde{g}(X, \phi Y) = 0 \end{cases}$$

for any vector fields X, Y tangent to $\widetilde{M}$ where I is the identity on sections of the tangent bundle $T\widetilde{M}$, ϕ is a tensor field of type $(1,1)$, η is a 1-form, ξ is a vector field and $\tilde{g}$ is a Riemannian metric on $\widetilde{M}$. Throughout the paper all manifolds and

Date: May 4, 2010.
2000 *Mathematics Subject Classification.* 53C40, 53C55, 53C12, 53C42.
Key words and phrases. semi-invariant $\xi^{\perp}$-submanifold, totally umbilical submanifold, totally geodesic leaves.
The third author was supported by Grant PN-II ID 398/2007-2010 (Romania).

maps are smooth. We denote by $\mathcal{F}(\widetilde{M})$ the algebra of the smooth functions on $\widetilde{M}$ and by $\Gamma(E)$ the $\mathcal{F}(\widetilde{M})$-module of the sections of a vector bundle E over $\widetilde{M}$.

The almost contact manifold $\widetilde{M}(\phi,\xi,\eta)$ is said to be *normal* if

$$N_\phi(X,Y)+2d\eta(X,Y)\xi=0$$

where

$$N_\phi(X,Y)=[\phi X,\phi Y]+\phi^2[X,Y]-\phi[\phi X,Y]-\phi[X,\phi Y],\quad X,Y\in\Gamma(T\widetilde{M})$$

is the Nijenhuis tensor field corresponding of the tensor field ϕ.

The *fundamental* 2*-form* Φ on $\widetilde{M}$ is defined by $\Phi(X,Y)=\tilde{g}(X,\phi Y)$.

In [8], the author studied hypersurfaces of an almost contact metric manifold $\widetilde{M}$ whose structure tensor fields satisfy the following relation

$$(\widetilde{\nabla}_X\phi)Y=\tilde{g}(\widetilde{\nabla}_{\phi X}\xi,Y)\xi-\eta(Y)\widetilde{\nabla}_{\phi X}\xi \tag{1}$$

where $\widetilde{\nabla}$ is the Levi-Civita connection of the metric tensor $\tilde{g}$. See also [6, 7]. For the sake of simplicity we say that a manifold $\widetilde{M}$ endowed with an almost contact metric structure satisfying (1) is a *generalized Quasi-Sasakian manifold*, in short G.Q.S. Define a $(1,1)$ type tensor field F by

$$FX=-\widetilde{\nabla}_X\xi. \tag{2}$$

Proposition 1. *If $\widetilde{M}$ is a G.Q.S manifold then any integral curve of the structure vector field ξ is a geodesic i.e. $\widetilde{\nabla}_\xi\xi=0$. Moreover $d\Phi=0$ if and only if ξ is a Killing vector field.*

Proof. The first assertion follows immediately from (1) with $X=Y=\xi$, and taking into account that $\eta(\widetilde{\nabla}_\xi\xi)=0$. Next, we deduce

$$3d\Phi(X,Y,Z)=\tilde{g}\big((\widetilde{\nabla}_X\phi)Z,Y\big)+\tilde{g}\big((\widetilde{\nabla}_Z\phi)Y,X\big)+\tilde{g}\big((\widetilde{\nabla}_Y\phi)X,Z\big)+$$
$$+\eta(X)\Big(\tilde{g}(Y,\widetilde{\nabla}_{\phi Z}\xi)+\tilde{g}(\phi Z,\widetilde{\nabla}_Y\xi)\Big)+\eta(Y)\Big(\tilde{g}(Z,\widetilde{\nabla}_{\phi X}\xi)+\tilde{g}(\phi X,\widetilde{\nabla}_Z\xi)\Big)+$$
$$+\eta(Z)\Big(\tilde{g}(X,\widetilde{\nabla}_{\phi Y}\xi)+\tilde{g}(\phi Y,\widetilde{\nabla}_X\xi)\Big).$$

If we suppose that ξ is Killing then, from the last equation, we obtain $d\Phi=0$.

Conversely, suppose that $d\Phi=0$. Taking into account the first part of the statement, for $X=\xi$, $\eta(Y)=\eta(Z)=0$, the last relation implies

$$\tilde{g}\big(Y,\widetilde{\nabla}_{\phi Z}\xi\big)+\tilde{g}\big(\phi Z,\widetilde{\nabla}_Y\xi\big)=0.$$

Finally, by replacing Z with ϕZ and Y by $Y-\eta(Y)\xi$ we deduce that ξ is a Killing vector field. $\square$

The next result can be obtained by direct calculation:

Proposition 2. *A G.Q.S manifold $\widetilde{M}$ is normal and*

$$\phi\circ F=F\circ\phi,\ \ F\xi=0,\ \ \eta\circ F=0,\ \ \widetilde{\nabla}_\xi\phi=0. \tag{3}$$

Remark 1. a) It is easy to see that on such manifold $\widetilde{M}$ the structure vector field ξ is not necessarily a Killing vector field i.e. $\widetilde{M}$ is not necessarily a K-contact manifold.
b) It is also interesting to pointed out that the following particular situations hold

1) $FX=-\phi X$ then $\widetilde{M}$ is Sasakian

2) $FX = -X + \eta(X)\xi$ then $\widetilde{M}$ is Kenmotsu
3) $FX = 0$ then $\widetilde{M}$ is cosymplectic
4) if ξ is a Killing vector field then $\widetilde{M}$ is a quasi-Sasakian manifold.

Now, let $\widetilde{M}$ be a G.Q.S manifold and consider an m-dimensional submanifold M, isometrically immersed in $\widetilde{M}$. Denote by g the induced metric on M and by ∇ its Levi-Civita connection. Let $\nabla^{\perp}$ and h be the normal connection induced by $\widetilde{\nabla}$ on the normal bundle $TM^{\perp}$ and the second fundamental form of M, respectively. Then one has the direct sum decomposition $T\widetilde{M} = TM \oplus TM^{\perp}$. Recall the Gauss and Weingarten formulae

$$\text{(G)}\quad \widetilde{\nabla}_X Y = \nabla_X Y + h(X,Y)$$

$$\text{(W)}\quad \widetilde{\nabla}_X N = -A_N X + \nabla^{\perp}_X N, \quad X, Y \in \Gamma(TM)$$

where A_N is the shape operator with respect to the normal section N and satisfies

$$\tilde{g}(h(X,Y),N) = g(A_N X, Y) \quad X, Y \in \Gamma(TM), \quad N \in \Gamma(TM^{\perp}).$$

The purpose of the present paper is to investigate the semi-invariant $\xi^{\perp}$-submanifolds in a G.Q.S manifold. More precisely, we suppose that the structure vector field ξ is orthogonal to the submanifold M. According to Bejancu [4] we say that M is a *semi-invariant $\xi^{\perp}$-submanifold* if there exist two orthogonal distributions, $\mathcal{D}$ and $\mathcal{D}^{\perp}$, in TM such that:

$$TM = \mathcal{D} \oplus \mathcal{D}^{\perp}, \ \phi\mathcal{D} = \mathcal{D}, \ \phi\mathcal{D}^{\perp} \subseteq TM^{\perp} \tag{4}$$

where $\oplus$ denotes the orthogonal sum. If $\mathcal{D}^{\perp} = \{0\}$ then M is an *invariant $\xi^{\perp}$-submanifold*. The normal bundle can also be decomposed as $TM^{\perp} = \phi\mathcal{D}^{\perp} \oplus \mu$, where $\phi\mu \subseteq \mu$. Hence μ contains ξ.

2. Integrability of distributions on a semi-invariant $\xi^{\perp}$-submanifold

Let M be a semi-invariant $\xi^{\perp}$-submanifold of a G.Q.S manifold $\widetilde{M}$. Denote by P and Q the projections of TM on $\mathcal{D}$ and $\mathcal{D}^{\perp}$ respectively, namely for any $X \in \Gamma(TM)$

$$X = PX + QX. \tag{5}$$

Moreover, for any $X \in \Gamma(TM)$ and $N \in \Gamma(TM^{\perp})$ we put

$$\phi X = tX + \omega X \tag{6}$$

$$\phi N = BN + CN \tag{7}$$

with $tX \in \Gamma(\mathcal{D})$, $BN \in \Gamma(TM)$ and $\omega X, CN \in \Gamma(TM^{\perp})$. We also consider, for $X \in \Gamma(TM)$, the decomposition

$$FX = \alpha X + \beta X, \quad \alpha X \in \Gamma(\mathcal{D}), \ \beta X \in \Gamma(TM^{\perp}). \tag{8}$$

The purpose of this section is to study the integrability of both distributions $\mathcal{D}$ and $\mathcal{D}^{\perp}$. With this scope in mind, we state first the following result.

Proposition 3. *Let M be a semi-invariant $\xi^{\perp}$-submanifold of a G.Q.S manifold $\widetilde{M}$. Then we have*

$$\begin{aligned} &a)\ (\nabla_X t)Y = A_{\omega Y}X + Bh(X,Y),\\ &b)\ (\nabla_X \omega)Y = Ch(X,Y) - h(X,tY) + g(FX,\phi Y)\xi, \quad X, Y \in \Gamma(TM). \end{aligned} \tag{9}$$

Proof. The statement follows immediately from (6)–(8). □

Taking into consideration the decomposition of $TM^{\perp}$, it can be easily proved:

Proposition 4. *Let M be a semi-invariant $\xi^{\perp}$-submanifold of a G.Q.S manifold $\widetilde{M}$. Then for any $N \in \Gamma(TM^{\perp})$ one has:*

a) $BN \in \mathcal{D}^{\perp}$,
b) $CN \in \mu$.

Proposition 5. *If M is a semi-invariant $\xi^{\perp}$-submanifold of a G.Q.S manifold $\widetilde{M}$ then*

$$A_{\omega Z}W = A_{\omega W}Z \tag{10}$$

for any $Z, W \in \Gamma(\mathcal{D}^{\perp})$.

The following two results give necessary and sufficient conditions for the integrability of the two distributions.

Theorem 1. *Let M be a semi-invariant $\xi^{\perp}$-submanifold of a G.Q.S manifold $\widetilde{M}$. Then the distribution $\mathcal{D}^{\perp}$ is integrable.*

Proof. Let $Z, W \in \Gamma(\mathcal{D}^{\perp})$. Then from (6), (9) and (10) we deduce that

$$t[Z, W] = A_{\omega Z}W - A_{\omega W}Z = 0.$$

Hence the conclusion. □

Theorem 2. *If M is a semi-invariant $\xi^{\perp}$-submanifold of a G.Q.S manifold $\widetilde{M}$ then the distribution $\mathcal{D}$ is integrable if and only if*

$$h(tX, Y) - h(X, tY) = (\mathcal{L}_{\xi}\tilde{g})(X, \phi Y)\ \xi, \quad X, Y \in \Gamma(\mathcal{D}). \tag{11}$$

Proof. The statement yields directly from (3) and (9)

$$\omega([X, Y]) = h(X, tY) - h(tX, Y) + (\mathcal{L}_{\xi}\tilde{g})(X, \phi Y)\ \xi.$$

□

Notice that the two results above are analogue those obtained in the Kenmotsu case in [12] and for the cosymplectic case in [14]. See also [10] when the submanifold is tangent to the structure vector field of the Sasakian manifold.

Moreover, from (8) we deduce

Proposition 6. *Let M be a $\xi^{\perp}$-semi-invariant submanifold of a G.Q.S manifold $\widetilde{M}$. Then*

$$A_{\xi}X = \alpha X, \quad \nabla^{\perp}_{X}\xi = -\beta X, \quad X \in \Gamma(TM). \tag{12}$$

Let now $\{e_i, \phi e_i, e_{2p+j}\}$, $i \in \{1, ..., p\}$, $j \in \{1, ..., q\}$ be an adapted orthonormal local frame on M, where $q = \dim \mathcal{D}^{\perp}$ and $2p = \dim \mathcal{D}$. One can state the following

Theorem 3. *If M is a $\xi^{\perp}$-semi-invariant submanifold of a G.Q.S manifold $\widetilde{M}$ one has*

$$\eta(H) = \frac{1}{m}\ \mathrm{trace}(A_{\xi}), \quad m = 2p + q.$$

Proof. Using a general formula for the mean curvature, e.g. $H = \frac{1}{m}\sum_{a=1}^{q} \operatorname{trace}\big(A_{\xi_a}\big)\xi_a$, where $\{\xi_1, \ldots, \xi_q\}$ is an orthonormal basis in $TM^{\perp}$, the conclusion holds by straightforward computations. □

In the case when the ambient space is a Kenmotsu manifold we retrieve the known result from [12, p. 614].

Corollary 1. *There does not exist a minimal semi-invariant $\xi^{\perp}$-submanifold of a Kenmotsu manifold.*

Also it is not difficult to prove:

Theorem 4. *Let M be a semi-invariant $\xi^{\perp}$-submanifold of a G.Q.S manifold $\widetilde{M}$. Then*

(1) *the distribution $\mathcal{D}$ is integrable and its leaves are totally geodesic in M if and only if $h(X, Y) \in \Gamma(\mu)$, where X, Y belong to $\mathcal{D}$;*
(2) *any leaf of the integrable distribution $\mathcal{D}^{\perp}$ is totally geodesic in M if and only if $h(X, Z) \in \Gamma(\mu)$ if $X \in \Gamma(\mathcal{D})$ and $Z \in \Gamma(\mathcal{D}^{\perp})$.*

Proof. Let us prove only the first statement. For any $Z \in \mathcal{D}^{\perp}$ we have

$$\tilde{g}\big(h(X,Y), \phi Z\big) = \tilde{g}\big(\widetilde{\nabla}_X Y, \phi Z\big) = -\tilde{g}\big(Y, \widetilde{\nabla}_X(\phi Z)\big) = \\ = -\tilde{g}\big(Y, (\widetilde{\nabla}_X \phi)Z\big) - \tilde{g}\big(\phi Y, \widetilde{\nabla}_X Z\big) = g\big(\nabla_X(\phi Y), Z\big).$$

Let M^* be a leaf of the integrable distribution $\mathcal{D}$ and h^* the second fundamental form of M^* in M.

For any $Z \in \Gamma(\mathcal{D}^{\perp})$ we have:

$$g(h^*(X,Y), Z) = \tilde{g}(\tilde{\nabla}_X tY, Z) = \tilde{g}((\tilde{\nabla}_X \varphi)Y + \varphi(\tilde{\nabla}_X Y), Z) = -\tilde{g}(h(X,Y), \varphi Z)$$

which proves that the leaf M^* of the integrable $\mathcal{D}$ is totally geodesic in M if and only if $h(X, Y) \in \Gamma(\mu)$.

Notice that the part (2) of the previous Theorem was obtained in the Kenmotsu case by Papaghiuc in [13, p. 115].

□

We end this section with the following

Corollary 2. *If the leaves of the integrable distribution $\mathcal{D}$ are totally geodesic in M then the structure vector field ξ is $\mathcal{D}$-Killing, that is $(\mathcal{L}_{\xi} g)(X, Y) = 0$, $X, Y \in \Gamma(\mathcal{D})$.*

3. Totally umbilical semi-invariant $\xi^{\perp}$-submanifolds

The main purpose of this section is to obtain a complete characterization of a totally umbilical semi-invariant $\xi^{\perp}$-submanifold of a G.Q.S manifold $\widetilde{M}$. Recall that for a totally umbilical submanifold we have

$$h(X, Y) = g(X, Y)H, \quad X, Y \in \Gamma(TM).$$

First we state:

Theorem 5. *An invariant $\xi^\perp$-submanifold M of a G.Q.S manifold is totally umbilical if and only if*

$$h(X,Y) = \frac{1}{m} g(X,Y)\mathrm{trace}\big(A_\xi\big)\xi. \tag{13}$$

Proof. If M is an invariant $\xi^\perp$-submanifold then for any $X, Y \in \Gamma(TM)$ we have $h(X, \phi Y) = \phi h(X,Y) - g(A_\xi \phi X, Y)\xi$. Let us consider an orthonormal frame $\{e_i, e_{p+i}\}$, $i = 1, \ldots, p$ on M; from the above relation one obtains that $\phi H = 0$. Again, since M is an invariant submanifold:

$$H = g(H,\xi)\xi = \frac{1}{m}\sum_{i=1}^{m} g(h(e_i,e_i),\xi)\xi = \frac{1}{m}\mathrm{trace}\big(A_\xi\big)\xi \tag{14}$$

and the proof is complete. □

Corollary 3. *A semi-invariant $\xi^\perp$-submanifold of a quasi-Sasakian manifold is minimal.*

The case of a semi-invariant $\xi^\perp$-submanifold in a G.Q.S manifold $\widetilde{M}$ is solved in the next Theorem.

Theorem 6. *Let M be a semi-invariant $\xi^\perp$-submanifold of a G.Q.S manifold $\widetilde{M}$ with* $\dim \mathcal{D}^\perp > 1$. *Then M is totally umbilical if and only if* (13) *holds.*

Proof. Let $X \in \Gamma(\mathcal{D})$ be a unit vector field and $N \in \Gamma(\mu) \setminus \mathrm{span}\{\xi\}$. By direct calculation it results that:

$$g(H,N) = g(h(X,X),N) = g(\widetilde{\nabla}_X \phi X - (\widetilde{\nabla}_X \phi)X, \phi N) = g(h(X,\phi X), \phi N) = 0$$

which proves that $H \in \phi\mathcal{D}^\perp \oplus \mathrm{span}\{\xi\}$.

For $Z, W \in \Gamma(\mathcal{D}^\perp)$, from (9) we derive $QA_{\phi Z}W = -g(Z,W)\phi H$ i.e.

$$g(Z,\phi H)g(W,\phi H) = g(Z,W)g(\phi H,\phi H). \tag{15}$$

If we take $Z = W$ orthogonal to ϕH, since $\dim \mathcal{D}^\perp > 1$, from the above relation we infer $\phi H = 0 \Rightarrow H \in \mathrm{span}\{\xi\}$. At this point the conclusion is straightforward.

Conversely, if (13) is supposed to be true, then we get (14) which together with (13) we deduce that M is totally umbilical. □

Let us remark that when $\widetilde{M}$ is a Kenmotsu manifold the result of the Theorem 6 was proved in [12].

Corollary 4. *Every $\xi^\perp$-hypersurface of a G.Q.S manifold $\widetilde{M}$ is totally umbilical.*

Proof. If M is a hypersurface then $TM^\perp = \mathrm{span}\{\xi\}$ that is $h(X,Y) \in \mathrm{span}\{\xi\}$. Next, from (14) it follows (13). □

In the particular case of a Kenmotsu manifold this result was obtained by Papaghiuc in [12, p. 617].

As a consequence of Theorem 6, we obtain

Theorem 7. *If M is a totally umbilical semi-invariant $\xi^\perp$-submanifold of a G.Q.S manifold $\widetilde{M}$ with* $\dim \mathcal{D}^\perp > 1$, *then M is a semi-invariant product.*

Here, by a semi-invariant product we mean a semi-invariant $\xi^{\perp}$-submanifold of $\widetilde{M}$ which can be locally written as a Riemannian product of a ϕ-invariant submanifold and a ϕ-anti-invariant submanifold of $\widetilde{M}$, both of them orthogonal to ξ.

Proof. From the definition of totally umbilical submanifold we have $h(X,Z) = 0$ for any $X \in \Gamma(\mathcal{D})$ and $Z \in \Gamma(\mathcal{D}^{\perp})$, so that, by b) of Theorem 4, the leaves of $\mathcal{D}^{\perp}$ are totally geodesic submanifolds of M. By Theorem 6, we have $h(X,Y) \in \operatorname{span}\{\xi\} \subset \mu$ for any $X, Y \in \mathcal{D}$. By virtue of a) of Theorem 1, this implies that the invariant distribution $\mathcal{D}$ is integrable and its integral manifolds are totally geodesic submanifolds of M. Therefore, we conclude that M is a semi-invariant product. $\square$

Without any restriction on the dimension of $\mathcal{D}^{\perp}$, we have the following

Theorem 8. *Let M be a totally umbilical semi-invariant $\xi^{\perp}$-submanifold of a G.Q.S manifold $\widetilde{M}$. If $\mathcal{D}$ is integrable, then each leaf of $\mathcal{D}$ is a totally geodesic submanifold of M.*

Proof. By using b) of Proposition 3, for any $X \in \Gamma(\mathcal{D})$, we have

$$\omega(\nabla_X X) = -g(X,X)CH - g(FX, \phi Y)\xi.$$

Since $CH \in \mu$ by b) of Lemma 4 and $\omega U \in \phi\mathcal{D}^{\perp}$ for any $U \in \Gamma(TM)$, from the above equation we deduce that $\omega(\nabla_X X) = 0$, or equivalently

$$\nabla_X X \in \mathcal{D}, \quad \forall X \in \Gamma(\mathcal{D}).$$

Replacing X by $X + Y$, we get $\nabla_X Y + \nabla_Y X \in \Gamma(\mathcal{D})$ for all $X, Y \in \Gamma(\mathcal{D})$. This condition, together with the integrability of $\mathcal{D}$, implies

$$\nabla_X Y \in \mathcal{D}, \quad \forall X, Y \in \Gamma(\mathcal{D}). \tag{16}$$

As $\mathcal{D}$ is integrable, Frobenius theorem ensures that M is foliated by leaves of $\mathcal{D}$. Combining this fact with (16), we conclude that the leaves of $\mathcal{D}$ are totally geodesic submanifolds of M. $\square$

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